

# Fokker-Planck type equations with Sobolev diffusion coefficients and BV drift coefficients

Dejun Luo\*

Institute of Applied Mathematics, Academy of Mathematics and Systems Science,  
Chinese Academy of Sciences, Beijing 100190, China

## Abstract

In this paper we give an affirmative answer to an open question mentioned in [Le Bris and Lions, Comm. Partial Differential Equations 33 (2008), 1272–1317], that is, we prove the well-posedness of the Fokker-Planck type equations with Sobolev diffusion coefficients and BV drift coefficients.

**Keywords:** Di Perna-Lions theory, Fokker-Planck equation, stochastic differential equation, BV regularity, commutator estimate

**MSC 2010:** 35Q84, 60H10

## 1 Introduction

The celebrated Di Perna-Lions theory, which was initiated in [7], claims that the well-posedness of the transport equation implies the existence and uniqueness of the quasi-invariant flow generated by a Sobolev vector field, provided its divergence is bounded. See [5] for the case of Gaussian measure as the reference measure. This theory has subsequently been extended to the case of BV vector fields by Ambrosio [1, 2], to the infinite dimensional Wiener space in [3, 9], and to the case of SDE in [16, 10], inspired by the work of Crippa and de Lellis [6]. In two recent papers [12, 13], Le Bris and Lions used again the ideas of the Di Perna-Lions theory to study the Fokker-Planck type equations, showing the existence and uniqueness of solutions when the coefficients have a certain Sobolev regularity (see also [11, 15] for some related results). In [14], the author studied the well-posedness of Fokker-Planck type equations on the Wiener space, mainly under the exponential integrability of the gradients and divergences of the coefficients. For the study of Fokker-Planck equations in the Hilbert space, see [4] and the references therein.

The present work is motivated by [13]. We consider the following Fokker-Planck equations

$$\partial_t p + \partial_i(p b_i) - \partial_{ij}^2(a_{ij} p) = 0, \quad (1.1)$$

where the coefficients  $b = (b_1, \dots, b_n)$  and  $a = (a_{ij})_{1 \leq i, j \leq n}$  have only weak regularity on the spatial variables, e.g. Sobolev or BV regularity. For the motivation of studying this kind of equations with irregular coefficients, see [13, Section 4]. The above equation (1.1) is closely related to the stochastic differential equation

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad X_0 = x,$$

---

\*Email: luodj@amss.ac.cn. This work is supported by the Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (No. 2008DP173182).

in which  $\sigma = (\sigma_{ik})_{1 \leq i \leq n, 1 \leq k \leq m}$  is a matrix-valued function such that  $a = \frac{1}{2}\sigma\sigma^*$  (here  $\sigma^*$  is the transpose of  $\sigma$ ), and  $W_t$  is an  $m$ -dimensional standard Brownian motion. In the following we always assume that the coefficient  $a$  has such a form, as in [13]. The adjoint equation of (1.1), called the backward Kolmogorov equation, reads as

$$\partial_t p - b_i \partial_i p - a_{ij} \partial_{ij}^2 p = 0. \quad (1.2)$$

We will mainly focus on the Fokker-Planck equation of divergence form (a notion introduced in (5.8) of [13]):

$$\partial_t p + \partial_i (p b_i) - \frac{1}{2} \partial_i (\sigma_{ik} \sigma_{jk} \partial_j p) = 0. \quad (1.3)$$

As pointed out in [13, Subsection 5.1], this equation is relatively easier to deal with, compared to the Fokker-Planck equation (1.1) and the backward Kolmogorov equation (1.2), since its second order term  $\partial_i (\sigma_{ik} \sigma_{jk} \partial_j p)$  has a self-adjoint form. To state the main result of this paper, we introduce some notations. For a locally integrable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we say that  $f$  has  $BV_{loc}$  regularity, denoted by  $f \in BV_{loc}(\mathbb{R}^n)$ , if for every  $i = 1, \dots, n$ , the distributional partial derivative  $\partial_i f$  of  $f$  is a locally finite Radon measure on  $\mathbb{R}^n$ .  $f^- := -(f \wedge 0)$  denotes the negative part of the function  $f$ . For a  $BV_{loc}$  vector field  $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we write  $D \cdot b = \sum_{i=1}^n \partial_i b_i$  for its divergence, which is still a Radon measure. If  $D \cdot b$  is absolutely continuous with respect to the Lebesgue measure  $dx$ , we denote by  $\text{div}(b)$  its Radon-Nikodym density. Our main result is the following

**Theorem 1.1.** *Assume that  $b$  and  $\sigma$  satisfy:*

$$\begin{aligned} b &\in (L^1([0, T], BV_{loc}(\mathbb{R}^n)))^n, \quad \frac{b}{1 + |x|} \in (L^1([0, T], L^1 + L^\infty(\mathbb{R}^n)))^n, \\ \text{div}(b) &\in L^1([0, T], L^1_{loc}(\mathbb{R}^n)), \quad [\text{div}(b)]^- \in L^1([0, T], L^\infty(\mathbb{R}^n)); \end{aligned} \quad (1.4)$$

and

$$\sigma \in (L^2([0, T], W^{1,2}_{loc}(\mathbb{R}^n)))^{n \times m}, \quad \frac{\sigma}{1 + |x|} \in (L^2([0, T], L^2 + L^\infty(\mathbb{R}^n)))^{n \times m}. \quad (1.5)$$

Then for each initial condition in  $L^2 \cap L^\infty$  (resp.  $L^1 \cap L^\infty$ ), the equation (1.3) has a unique solution in the space

$$\{p \in L^\infty([0, T], L^2 \cap L^\infty) \text{ (resp. } L^\infty([0, T], L^1 \cap L^\infty)), \sigma^* \nabla p \in L^2([0, T], L^2)\}.$$

This theorem gives a positive answer to the open question raised in [13, Remark 12] on page 1299. It also generalizes [11, Theorem 1.4], since the diffusion coefficient  $a$  in [11] is assumed to be independent on the spatial variables. The proof of Theorem 1.1 will be provided in the next section. As in [13, Section 7], we can adapt the above result to the other Fokker-Planck type equations (1.1) and (1.2), by imposing suitable conditions on  $b^\sigma = b - \frac{1}{2} \text{div}(\sigma\sigma^*)$ . Finally we give a brief discussion on the reason why we cannot further generalize it to the case where  $\sigma$  has only BV regularity, see Remark 2.6.

## 2 Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. We consider the Fokker-Planck equation of divergence form

$$\partial_t p + \partial_i (p b_i) - \frac{1}{2} \partial_i (\sigma_{ik} \sigma_{jk} \partial_j p) = 0. \quad (2.1)$$

First we give the mathematical meaning of the above equation. For a given initial condition  $p_0 \in L^2 \cap L^\infty$  (resp.  $L^1 \cap L^\infty$ ), a function  $p \in L^\infty([0, T], L^2 \cap L^\infty)$  (resp.  $L^\infty([0, T], L^1 \cap L^\infty)$ ) satisfying  $\sigma^* \nabla p \in L^2([0, T], L^2)$  is called a weak solution to (2.1) if for all  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^n)$ , it holds

$$\int_0^T \int_{\mathbb{R}^n} p \partial_t \varphi \, dx dt + \int_{\mathbb{R}^n} p_0 \varphi(0, \cdot) \, dx = - \int_0^T \int_{\mathbb{R}^n} p \langle b, \nabla \varphi \rangle \, dx dt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \langle \sigma^* \nabla p, \sigma^* \nabla \varphi \rangle \, dx dt.$$

The existence of solutions to the above equation is the easier part, see for instance the beginning of [13, Subsection 5.4] for the case  $p_0 \in L^2 \cap L^\infty$ , and [13, Subsection 6.1.2] for the case  $p_0 \in L^1 \cap L^\infty$ . Hence in the following we focus on the uniqueness part of Theorem 1.1 and follow the ideas in [13, Subsection 5.3]. The main difference is that a single convolution kernel is not enough to achieve our result, instead, we need a family of kernels as in the proof of [1, Theorem 3.5], see also [2, Theorem 5.1].

Let

$$\mathcal{K} = \left\{ \rho \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+) : \text{supp}(\rho) \subset B(1), \int_{\mathbb{R}^n} \rho \, dx = 1 \right\}$$

be the family of candidate convolution kernels, where  $B(1)$  is the open unit ball centered at the origin 0. Take  $\rho \in \mathcal{K}$  and define  $\rho_\varepsilon = \varepsilon^{-n} \rho(\varepsilon^{-1} \cdot)$  for  $\varepsilon > 0$ . We regularize the equation (2.1) in the spatial variables:

$$\partial_t(\rho_\varepsilon * p) + \rho_\varepsilon * \partial_i(p b_i) - \frac{1}{2} \rho_\varepsilon * \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p) = 0. \quad (2.2)$$

We denote by  $p_\varepsilon = \rho_\varepsilon * p$  and introduce the notation

$$[\rho_\varepsilon, c](f) = \rho_\varepsilon * (cf) - c(\rho_\varepsilon * f)$$

for a differential operator  $c$ . Note that  $c$  can also be a real valued function. Using this notation, we have

$$\begin{aligned} \rho_\varepsilon * \partial_i(p b_i) &= \rho_\varepsilon * (\text{div}(b)p) + \rho_\varepsilon * (b_i \partial_i p) \\ &= [\rho_\varepsilon, \text{div}(b)](p) + [\rho_\varepsilon, b_i \partial_i](p) + \partial_i(b_i p_\varepsilon) \\ &= Q_{1,\varepsilon} + Q_{2,\varepsilon} + \partial_i(b_i p_\varepsilon), \end{aligned} \quad (2.3)$$

where we have defined

$$Q_{1,\varepsilon} = [\rho_\varepsilon, \text{div}(b)](p) \quad \text{and} \quad Q_{2,\varepsilon} = [\rho_\varepsilon, b_i \partial_i](p). \quad (2.4)$$

In fact, it is the term  $Q_{2,\varepsilon}$  that causes the trouble in the BV situation and marks the difference between the present work and [13].

Next we have

$$\begin{aligned} \rho_\varepsilon * \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p) &= \rho_\varepsilon * ((\partial_i \sigma_{ik}) \sigma_{jk} \partial_j p + \sigma_{ik} \partial_i(\sigma_{jk} \partial_j p)) \\ &= [\rho_\varepsilon, \partial_i \sigma_{ik}](\sigma_{jk} \partial_j p) + (\partial_i \sigma_{ik}) \rho_\varepsilon * (\sigma_{jk} \partial_j p) \\ &\quad + [\rho_\varepsilon, \sigma_{ik} \partial_i](\sigma_{jk} \partial_j p) + \sigma_{ik} \partial_i(\rho_\varepsilon * (\sigma_{jk} \partial_j p)). \end{aligned}$$

Define

$$S_\varepsilon = [\rho_\varepsilon, \partial_i \sigma_{ik}](\sigma_{jk} \partial_j p) \quad \text{and} \quad T_\varepsilon = [\rho_\varepsilon, \sigma_{ik} \partial_i](\sigma_{jk} \partial_j p). \quad (2.5)$$

We remark that the term  $S_\varepsilon$  prevents us from extending the results to the case where the diffusion coefficient  $\sigma$  has only BV regularity, see Remark 2.6 for more details. Now we have

$$\rho_\varepsilon * \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p) = S_\varepsilon + T_\varepsilon + \partial_i(\sigma_{ik} \rho_\varepsilon * (\sigma_{jk} \partial_j p)).$$

If we denote by

$$R_{k,\varepsilon} = [\rho_\varepsilon, \sigma_{jk} \partial_j](p), \quad k = 1, \dots, m; \quad (2.6)$$

then

$$\rho_\varepsilon * (\sigma_{jk} \partial_j p) = R_{k,\varepsilon} + \sigma_{jk} \partial_j p_\varepsilon.$$

Summing up the above discussions, we arrive at the equality (5.17) in [13]:

$$\rho_\varepsilon * \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p) = S_\varepsilon + T_\varepsilon + \partial_i(\sigma_{ik} R_{k,\varepsilon}) + \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p_\varepsilon). \quad (2.7)$$

Combining (2.2), (2.3) and (2.7), we get an equation of  $p_\varepsilon$  similar to (2.1), but with some error terms on the right hand side:

$$\partial_t p_\varepsilon + \partial_i(p_\varepsilon b_i) - \frac{1}{2} \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p_\varepsilon) = -Q_{1,\varepsilon} - Q_{2,\varepsilon} + \frac{1}{2}(\partial_i(\sigma_{ik} R_{k,\varepsilon}) + S_\varepsilon + T_\varepsilon). \quad (2.8)$$

Now we need the classical commutator estimate in the Di Perna-Lions theory (see [7, Lemma II.1] or [13, Lemma 1]). We include it here for the sake of the readers' convenience.

**Lemma 2.1 (Commutator estimate I).** *For  $r, \alpha, r_1, \alpha_1 \geq 1$ , set  $\frac{1}{\beta} = \frac{1}{r} + \frac{1}{\alpha}$  and  $\frac{1}{\beta_1} = \frac{1}{r_1} + \frac{1}{\alpha_1}$ . Let  $f \in L^{r_1}([0, T], L^r_{loc}(\mathbb{R}^n))$ ,  $g \in L^{\alpha_1}([0, T], L^\alpha_{loc}(\mathbb{R}^n))$  and  $c \in (L^{\alpha_1}([0, T], W^{1,\alpha}_{loc}(\mathbb{R}^n)))^n$ . Then as  $\varepsilon \rightarrow 0$ ,*

$$[\rho_\varepsilon, c_i \partial_i](f) \rightarrow 0 \quad \text{in } L^{\beta_1}([0, T], L^\beta_{loc}(\mathbb{R}^n)), \quad (2.9)$$

and

$$[\rho_\varepsilon, g](f) \rightarrow 0 \quad \text{in } L^{\beta_1}([0, T], L^\beta_{loc}(\mathbb{R}^n)). \quad (2.10)$$

**Remark 2.2.** We observe that under the assumptions of the above lemma, for any fixed  $\varepsilon > 0$ , the commutator  $[\rho_\varepsilon, c_i \partial_i](f)$  belongs to the space  $L^{\beta_1}([0, T], W^{1,\beta}_{loc}(\mathbb{R}^n))$ , i.e., it has the first order Sobolev regularity with respect to the spatial variables. This can be seen from its expression (see (3.8) in [1] or [8, Lemma 2.5]):

$$\begin{aligned} [\rho_\varepsilon, c_i \partial_i](f)(x) &= -[(f \operatorname{div}(c)) * \rho_\varepsilon](x) + \int_{\mathbb{R}^n} f(y) \langle c(y) - c(x), (\nabla \rho_\varepsilon)(x - y) \rangle dy \\ &= -[(f \operatorname{div}(c)) * \rho_\varepsilon](x) + [(fc) * (\nabla \rho_\varepsilon)](x) - \langle c, f * (\nabla \rho_\varepsilon) \rangle(x). \end{aligned}$$

Indeed, the three terms  $(f \operatorname{div}(c)) * \rho_\varepsilon$ ,  $(fc) * (\nabla \rho_\varepsilon)$  and  $f * (\nabla \rho_\varepsilon)$  are smooth, therefore the assertion follows from the fact that  $c \in (L^{\alpha_1}([0, T], W^{1,\alpha}_{loc}(\mathbb{R}^n)))^n$ .  $\square$

We concentrate in the following on the  $L^2$ -theory, that is, we prove the uniqueness of solutions to (2.1) in the space

$$X_2 = \{p \in L^\infty([0, T], L^2 \cap L^\infty); \sigma^* \nabla p \in L^2([0, T], L^2)\}.$$

For the  $L^1$ -theory, the same argument as in [13, Subsection 6.1.2] works.

Now we consider the error terms in (2.8). Using Lemma 2.1, we see that for the time-dependent vector field  $b \in (L^1([0, T], \operatorname{BV}_{loc}(\mathbb{R}^n)))^n$  satisfying  $\operatorname{div}(b) \in L^1([0, T], L^1_{loc}(\mathbb{R}^n))$ , the first error term

$$Q_{1,\varepsilon} = [\rho_\varepsilon, \operatorname{div}(b)](p) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^1([0, T], L^1_{loc}(\mathbb{R}^n)), \quad (2.11)$$

since  $p \in L^\infty([0, T], L^\infty(\mathbb{R}^n))$ .

The estimate of the commutator  $Q_{2,\varepsilon}$  cannot be obtained from Lemma 2.1, instead, we will rely on the work of Ambrosio (see [1, Theorem 3.2] or [2, Section 5]). We first introduce some notations. Let  $Db_t = (\partial_j b_i(t))_{1 \leq i, j \leq n}$  be the ‘‘Jacobi’’ matrix of distributional derivatives of  $b_t$ ,

whose entries are locally finite Radon measures since  $b_t \in \text{BV}_{loc}(\mathbb{R}^n)$ . Denote by  $|Db_t|$  its total variation. Let

$$Db_t = D^a b_t + D^s b_t$$

be the Lebesgue decomposition of  $Db_t$  into absolutely continuous and singular part with respect to the Lebesgue measure  $dx$ . Define the matrix valued function  $M : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n$  as the Radon-Nikodym derivative of  $Db$  with respect to  $|Db|$ . Here the measure  $|Db|$  on  $[0, T] \times \mathbb{R}^n$  is defined as

$$\int \varphi(t, x) d|Db|(t, x) = \int_0^T \int_{\mathbb{R}^n} \varphi(t, x) d|Db_t|(x) dt, \quad \text{for all } \varphi \in C_c([0, T] \times \mathbb{R}^n).$$

$Db$ ,  $|D^a b|$  and  $|D^s b|$  are defined similarly. We remark that Ambrosio defined in [1] the matrix  $\tilde{M}_t$  to be the Radon-Nikodym derivative of  $D^s b$  with respect to  $|D^s b|$ ; however, when restricted on the support of  $|D^s b|$ , we have  $M_t(x) = \tilde{M}_t(x)$  for  $|D^s b|$ -a.e.  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

**Lemma 2.3 (Commutator estimate II).** *For any compact set  $K \subset (0, T) \times \mathbb{R}^n$ , we have*

$$\overline{\lim}_{\varepsilon \downarrow 0} \int_K |Q_{2,\varepsilon}| dx dt \leq \|p\|_\infty \int_K \Lambda(M_t(x), \rho) d|D^s b|(t, x) + \|p\|_\infty (n + I(\rho)) |D^a b|(K) \quad (2.12)$$

and

$$\overline{\lim}_{\varepsilon \downarrow 0} \int_K |Q_{2,\varepsilon}| dx dt \leq \|p\|_\infty I(\rho) |D^s b|(K), \quad (2.13)$$

where for a matrix  $M$  and  $\rho \in C_c^\infty(\mathbb{R}^n)$ ,

$$\Lambda(M, \rho) = \int_{\mathbb{R}^n} |\langle Mz, \nabla \rho(z) \rangle| dz, \quad I(\rho) = \int_{\mathbb{R}^n} |z| \cdot |\nabla \rho(z)| dz.$$

Now we turn to the error terms concerning the diffusion coefficient  $\sigma$ . The arguments are similar to those in [13, Subsection 5.3]. Applying again Lemma 2.1, we have

$$R_{k,\varepsilon} = [\rho_\varepsilon, \sigma_{jk} \partial_j](p) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^2([0, T], L_{loc}^2(\mathbb{R}^n)), \quad (2.14)$$

due to the facts that  $\sigma_{jk} \in L^2([0, T], W_{loc}^{1,2}(\mathbb{R}^n))$  and  $p \in L^\infty([0, T], L^\infty(\mathbb{R}^n))$ . Next, since both  $\partial_i \sigma_{ik}$  and  $\sigma_{jk} \partial_j p$  belong to  $L^2([0, T], L_{loc}^2(\mathbb{R}^n))$ ,

$$S_\varepsilon = [\rho_\varepsilon, \partial_i \sigma_{ik}](\sigma_{jk} \partial_j p) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^1([0, T], L_{loc}^1(\mathbb{R}^n)). \quad (2.15)$$

Finally, as  $\sigma_{ik} \in L^2([0, T], W_{loc}^{1,2}(\mathbb{R}^n))$ ,  $i = 1, \dots, n$ , we have

$$T_\varepsilon = [\rho_\varepsilon, \sigma_{ik} \partial_i](\sigma_{jk} \partial_j p) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^1([0, T], L_{loc}^1(\mathbb{R}^n)). \quad (2.16)$$

We denote by

$$U_\varepsilon = -Q_{1,\varepsilon} + \frac{1}{2}(S_\varepsilon + T_\varepsilon);$$

then the estimates (2.11), (2.15) and (2.16) lead to

$$U_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^1([0, T], L_{loc}^1(\mathbb{R}^n)). \quad (2.17)$$

The equation (2.8) can be rewritten as follows:

$$\partial_t p_\varepsilon + \partial_i(p_\varepsilon b_i) - \frac{1}{2} \partial_i(\sigma_{ik} \sigma_{jk} \partial_j p_\varepsilon) = U_\varepsilon - Q_{2,\varepsilon} + \frac{1}{2} \partial_i(\sigma_{ik} R_{k,\varepsilon}). \quad (2.18)$$

Notice that  $p_\varepsilon$  is smooth with respect to the spatial variable. By Remark 2.2, the commutator  $R_{k,\varepsilon} \in L^2([0, T], W_{loc}^{1,2}(\mathbb{R}^n))$ , which together with the regularity assumptions on the coefficients  $b$  and  $\sigma$  tells us that  $\partial_t p_\varepsilon \in L_{loc}^1([0, T] \times \mathbb{R}^n)$ . Therefore  $p_\varepsilon \in W_{loc}^{1,1}([0, T] \times \mathbb{R}^n)$  and we can apply the standard chain rule in Sobolev spaces. For  $\beta \in C^2(\mathbb{R})$ , one has

$$\begin{aligned} & \partial_t \beta(p_\varepsilon) + \partial_i(\beta(p_\varepsilon)b_i) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p_\varepsilon)) \\ &= \beta'(p_\varepsilon)\partial_t p_\varepsilon + b_i\beta'(p_\varepsilon)\partial_i p_\varepsilon + \beta(p_\varepsilon)\operatorname{div}(b) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\beta'(p_\varepsilon)\partial_j p_\varepsilon) \\ &= \beta'(p_\varepsilon)\partial_t p_\varepsilon + \beta'(p_\varepsilon)\partial_i(p_\varepsilon b_i) + (\beta(p_\varepsilon) - p_\varepsilon\beta'(p_\varepsilon))\operatorname{div}(b) - \frac{1}{2}\beta'(p_\varepsilon)\partial_i(\sigma_{ik}\sigma_{jk}\partial_j p_\varepsilon) \\ &\quad - \frac{1}{2}(\sigma_{ik}\sigma_{jk}\partial_j p_\varepsilon)\beta''(p_\varepsilon)\partial_i p_\varepsilon \\ &= \beta'(p_\varepsilon)\left[\partial_t p_\varepsilon + \partial_i(p_\varepsilon b_i) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j p_\varepsilon)\right] + (\beta(p_\varepsilon) - p_\varepsilon\beta'(p_\varepsilon))\operatorname{div}(b) - \frac{1}{2}\beta''(p_\varepsilon)|\sigma^*\nabla p_\varepsilon|^2. \end{aligned}$$

By (2.18), we obtain

$$\begin{aligned} & \partial_t \beta(p_\varepsilon) + \partial_i(\beta(p_\varepsilon)b_i) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p_\varepsilon)) - (\beta(p_\varepsilon) - p_\varepsilon\beta'(p_\varepsilon))\operatorname{div}(b) + \frac{1}{2}\beta''(p_\varepsilon)|\sigma^*\nabla p_\varepsilon|^2 \\ &= \beta'(p_\varepsilon)\left[U_\varepsilon - Q_{2,\varepsilon} + \frac{1}{2}\partial_i(\sigma_{ik}R_{k,\varepsilon})\right]. \end{aligned} \quad (2.19)$$

In order to prove the uniqueness of solutions to (2.1), we shall use the technique of renormalized solutions, a notion which was introduced by Di Perna and Lions in [7, Section II.3] (see also [2, Definition 4.1] and [11, Definition 4.9]).

**Definition 2.4 (Renormalized solution).** *Let  $\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^m \otimes \mathbb{R}^n$  and  $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be such that (i)  $\sigma \in L_{loc}^2([0, T] \times \mathbb{R}^n)$ , (ii)  $b, \operatorname{div}(b) \in L_{loc}^1([0, T] \times \mathbb{R}^n)$ . We say that a solution  $p$  to (2.1) is a renormalized solution if for any  $\beta \in C^2(\mathbb{R})$ , the following equation holds in the distributional sense:*

$$\partial_t \beta(p) + \partial_i(\beta(p)b_i) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p)) - (\beta(p) - p\beta'(p))\operatorname{div}(b) + \frac{1}{2}\beta''(p)|\sigma^*\nabla p|^2 = 0. \quad (2.20)$$

Now we show that any weak solution in the space  $X_2$  of (2.1) is renormalizable, provided the conditions of Theorem 1.1 are satisfied. The main idea of the arguments are similar to the proof of [1, Theorem 3.5] (see in particular Step 3 therein).

**Theorem 2.5 (Renormalization property).** *Under the conditions of Theorem 1.1, any weak solution*

$$p \in X_2 = \{p \in L^\infty([0, T], L^2 \cap L^\infty); \sigma^*\nabla p \in L^2([0, T], L^2)\}$$

*of (2.1) is also a renormalized solution.*

**Proof.** We have to show that as  $\varepsilon \downarrow 0$ , all the terms on the left hand side of the equation (2.19) converge in the distributional sense to the corresponding ones in (2.20), while the limit of the right hand side is 0. We split the proof into two steps: in the first step we show the convergences of all the terms except the one involving  $Q_{2,\varepsilon}$ , while in the second step we focus on the term  $\beta'(p_\varepsilon)Q_{2,\varepsilon}$ .

**Step 1.** Since  $p$  is essentially bounded and  $\beta \in C^2(\mathbb{R})$ ,  $\beta$  and its derivatives are uniformly continuous on the interval  $I_p = [-\|p\|_\infty, \|p\|_\infty]$ . Notice also that  $\|p_\varepsilon\|_\infty \leq \|p\|_\infty$ . As the proofs are similar, we only illustrate the convergences of

$$\partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p_\varepsilon)) \rightarrow \partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p)) \quad \text{and} \quad \beta'(p_\varepsilon)\partial_i(\sigma_{ik}R_{k,\varepsilon}) \rightarrow 0$$

as  $\varepsilon \downarrow 0$  in the distributional sense.

First for any  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^n)$ , we have by the integration by parts formula that

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^n} \varphi \partial_i (\sigma_{ik} \sigma_{jk} \partial_j \beta(p_\varepsilon)) \, dx dt - \int_0^T \int_{\mathbb{R}^n} \varphi \partial_i (\sigma_{ik} \sigma_{jk} \partial_j \beta(p)) \, dx dt \right| \\ & \leq \int_0^T \int_{\mathbb{R}^n} |\beta'(p_\varepsilon) \langle \sigma^* \nabla \varphi, \sigma^* \nabla p_\varepsilon \rangle - \beta'(p) \langle \sigma^* \nabla \varphi, \sigma^* \nabla p \rangle| \, dx dt. \end{aligned}$$

The above quantity is dominated by the sum of

$$J_{1,\varepsilon} := \|\beta'(p_\varepsilon)\|_\infty \|\nabla \varphi\|_\infty \int_K |\sigma| \cdot |\sigma^* \nabla p_\varepsilon - \sigma^* \nabla p| \, dx dt$$

and

$$J_{2,\varepsilon} := \|\nabla \varphi\|_\infty \int_K |\beta'(p_\varepsilon) - \beta'(p)| \cdot |\sigma| \cdot |\sigma^* \nabla p| \, dx dt,$$

where the compact set  $K := \text{supp}(\varphi) \subset [0, T] \times \mathbb{R}^n$  is the support of  $\varphi$ . By Cauchy's inequality, we have

$$J_{1,\varepsilon} \leq C \left( \int_K |\sigma|^2 \, dx dt \right)^{\frac{1}{2}} \left( \int_K |\sigma^* \nabla p_\varepsilon - \sigma^* \nabla p|^2 \, dx dt \right)^{\frac{1}{2}} \rightarrow 0$$

as  $\varepsilon \downarrow 0$ . In this paper  $C$  denotes the constants whose values have no importance and may change from line to line. Next since  $p \in L^\infty([0, T], L^2(\mathbb{R}^n)) \subset L^2([0, T], L^2(\mathbb{R}^n))$ ,  $p_\varepsilon$  tends to  $p$  in the latter space as  $\varepsilon \downarrow 0$ ; consequently,  $\beta'(p_\varepsilon)$  converges to  $\beta'(p)$  in measure. Moreover,

$$|\beta'(p_\varepsilon) - \beta'(p)| \cdot |\sigma| \cdot |\sigma^* \nabla p| \leq (\|\beta'(p_\varepsilon)\|_\infty + \|\beta'(p)\|_\infty) |\sigma| \cdot |\sigma^* \nabla p| \leq C |\sigma| \cdot |\sigma^* \nabla p|$$

which is integrable on  $K$ . Therefore the dominated convergence theorem tells us that

$$\lim_{\varepsilon \downarrow 0} J_{2,\varepsilon} = 0.$$

To sum up, we conclude that when  $\varepsilon \downarrow 0$ ,  $\partial_i (\sigma_{ik} \sigma_{jk} \partial_j \beta(p_\varepsilon))$  converges to  $\partial_i (\sigma_{ik} \sigma_{jk} \partial_j \beta(p))$  in the distributional sense.

Now we consider the limit  $\beta'(p_\varepsilon) \partial_i (\sigma_{ik} R_{k,\varepsilon}) \rightarrow 0$ . For  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^n)$ , again by integrating by parts, one has

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} \varphi \beta'(p_\varepsilon) \partial_i (\sigma_{ik} R_{k,\varepsilon}) \, dx dt \\ & = - \int_0^T \int_{\mathbb{R}^n} \beta'(p_\varepsilon) \langle \sigma^* \nabla \varphi, R_\varepsilon \rangle \, dx dt - \int_0^T \int_{\mathbb{R}^n} \varphi \beta''(p_\varepsilon) \langle \sigma^* \nabla p_\varepsilon, R_\varepsilon \rangle \, dx dt \\ & =: -J_{3,\varepsilon} - J_{4,\varepsilon}, \end{aligned}$$

where  $R_\varepsilon = (R_{1,\varepsilon}, \dots, R_{m,\varepsilon})$ . Note that

$$\begin{aligned} |J_{3,\varepsilon}| & \leq \|\beta'(p_\varepsilon)\|_\infty \|\nabla \varphi\|_\infty \int_K |\sigma| \cdot |R_\varepsilon| \, dx dt \\ & \leq C \left( \int_K |\sigma|^2 \, dx dt \right)^{\frac{1}{2}} \left( \int_K |R_\varepsilon|^2 \, dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore by (2.14), we get  $\lim_{\varepsilon \downarrow 0} J_{3,\varepsilon} = 0$ . Since  $\beta''(p_\varepsilon)$  and  $\|\sigma^* \nabla p_\varepsilon\|_{L^2([0, T] \times \mathbb{R}^n)}$  are bounded, uniformly in  $\varepsilon > 0$ , we can show that  $\lim_{\varepsilon \downarrow 0} J_{4,\varepsilon} = 0$  in the same way.

Summing up these discussions, we conclude that as  $\varepsilon \downarrow 0$ , all the terms in (2.19), except  $-\beta'(p_\varepsilon)Q_{2,\varepsilon}$ , converge in the distributional sense. We define the “defect” measure

$$\mu := \partial_t \beta(p) + \partial_i(\beta(p)b_i) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p)) - (\beta(p) - p\beta'(p))\operatorname{div}(b) + \frac{1}{2}\beta''(p)|\sigma^*\nabla p|^2 \quad (2.21)$$

(notice that  $\beta'(p_\varepsilon)[U_\varepsilon + \frac{1}{2}\partial_i(\sigma_{ik}R_{k,\varepsilon})] \rightarrow 0$  as  $\varepsilon$  tends to 0).

**Step 2.** Now we deal with the term  $-\beta'(p_\varepsilon)Q_{2,\varepsilon}$ . Let  $Q_\rho$  be one of the weak limit points of  $|\beta'(p_\varepsilon)Q_{2,\varepsilon}|$  in the sense of measure (by Lemma 2.3, such an accumulating point exists since  $|\beta'(p_\varepsilon)Q_{2,\varepsilon}|$  is bounded in  $L^1_{loc}$ ). The measure  $Q_\rho$  depends on the convolution kernel  $\rho$ , but the “defect” measure  $\mu$  defined in (2.21) is independent of  $\rho$  and satisfies  $|\mu| \leq Q_\rho$ . Thus we get

$$|\mu| \leq Q_\rho \quad \text{for all } \rho \in \mathcal{K}. \quad (2.22)$$

We deduce from (2.13) that  $Q_\rho$  is absolutely continuous with respect to  $|D^s b|$ , which together with (2.12) gives us

$$Q_\rho \leq \|p\|_\infty \Lambda(M(\cdot), \rho) |D^s b|.$$

Thus by (2.22) we obtain

$$|\mu| \leq \|p\|_\infty \Lambda(M(\cdot), \rho) |D^s b|, \quad \text{for all } \rho \in \mathcal{K}.$$

Denote by  $g$  the Radon-Nikodym density of  $|\mu|$  with respect to  $|D^s b|$ . Then for each  $\rho \in \mathcal{K}$ ,

$$g(t, x) \leq \|p\|_\infty \Lambda(M_t(x), \rho) \quad \text{for } |D^s b| \text{-a.e. } (t, x).$$

Let  $\mathcal{D}$  be a countable dense subset of  $\mathcal{K}$  with respect to the norm  $W^{1,1}(B(1))$ . We have

$$g(t, x) \leq \|p\|_\infty \inf_{\rho \in \mathcal{D}} \Lambda(M_t(x), \rho) \quad \text{for } |D^s b| \text{-a.e. } (t, x). \quad (2.23)$$

For  $|D^s b|$ -a.e.  $(t, x)$  fixed, we deduce from the definition of  $\Lambda(M_t(x), \rho)$  that the mapping  $\mathcal{K} \ni \rho \rightarrow \Lambda(M_t(x), \rho)$  is continuous with respect to the  $W^{1,1}(B(1))$  norm. Therefore

$$\inf_{\rho \in \mathcal{D}} \Lambda(M_t(x), \rho) = \inf_{\rho \in \mathcal{K}} \Lambda(M_t(x), \rho).$$

Now by Alberti’s rank one structure of  $M_t(x)$  (cf. [1, Theorem 2.3]), we conclude that the above infimum is 0 for  $|D^s b|$ -a.e.  $(t, x) \in [0, T] \times \mathbb{R}^n$  (see [1, Lemma 3.3]). Therefore  $g = 0$   $|D^s b|$ -a.e. by (2.23). As a result, the measure  $\mu = 0$ . By (2.21), we arrive at

$$\partial_t \beta(p) + \partial_i(\beta(p)b_i) - \frac{1}{2}\partial_i(\sigma_{ik}\sigma_{jk}\partial_j\beta(p)) - (\beta(p) - p\beta'(p))\operatorname{div}(b) + \frac{1}{2}\beta''(p)|\sigma^*\nabla p|^2 = 0. \quad (2.24)$$

The proof is complete.  $\square$

Finally we are ready to prove the uniqueness of solutions to (1.3).

**Proof of Theorem 1.1.** Since the equation (1.3) is linear, it is enough to show that if  $p$  is a weak solution such that  $p(0) \equiv 0$ , then  $p(t) \equiv 0$  for all  $t \geq 0$ . By Theorem 2.5, we know that for any  $\beta \in C^2(\mathbb{R})$ , (2.24) holds in the distributional sense. If we choose  $\beta(s) = s^2$  for  $s \in \mathbb{R}$ , then it becomes

$$\partial_t(p^2) + \partial_i(p^2 b_i) - \partial_i(\sigma_{ik}\sigma_{jk}p \partial_j p) + p^2 \operatorname{div}(b) \leq 0. \quad (2.25)$$

We take  $\phi \in C_c^\infty(\mathbb{R}^n, [0, 1])$  satisfying  $\phi|_{B(1)} \equiv 1$  and  $\operatorname{supp}(\phi) \subset B(2)$ . Define a nonnegative smooth cut-off function  $\phi_R = \phi(\frac{\cdot}{R})$  for  $R > 0$ ; then  $\nabla \phi_R = \frac{1}{R} \nabla \phi(\frac{\cdot}{R})$ . To simplify the notations,



we will write  $\int_{\mathbb{R}^n} f$  for the integral of the function  $f$  on  $\mathbb{R}^n$  with respect to the Lebesgue measure. Multiplying the inequality (2.25) with  $\phi_R$  and integrating by parts on  $\mathbb{R}^n$ , we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^n} p^2 \phi_R \leq \int_{\mathbb{R}^n} p^2 \langle b, \nabla \phi_R \rangle - \int_{\mathbb{R}^n} p \langle \sigma^* \nabla \phi_R, \sigma^* \nabla p \rangle - \int_{\mathbb{R}^n} p^2 \phi_R \operatorname{div}(b). \quad (2.26)$$

Now we estimate the three terms on the right hand side. First

$$\begin{aligned} \left| \int_{\mathbb{R}^n} p^2 \langle b, \nabla \phi_R \rangle \right| &= \left| \int_{\{R \leq |x| \leq 2R\}} p^2 \langle b, \frac{1}{R} \nabla \phi(\frac{\cdot}{R}) \rangle \right| \\ &\leq C \|\nabla \phi\|_{L^\infty} \int_{\{|x| \geq R\}} p^2 \frac{|b|}{1+|x|}. \end{aligned}$$

Since  $\frac{b}{1+|x|} \in (L^1([0, T], L^1 + L^\infty(\mathbb{R}^n)))^n$ , there are two vector fields  $b_1, b_2$  such that  $b = b_1 + b_2$  and  $\frac{|b_1|}{1+|x|} \in L^1([0, T], L^1(\mathbb{R}^n))$ ,  $\frac{|b_2|}{1+|x|} \in L^1([0, T], L^\infty(\mathbb{R}^n))$ . Then we have

$$\left| \int_{\mathbb{R}^n} p^2 \langle b, \nabla \phi_R \rangle \right| \leq C \|p\|_{L^\infty([0, T], L^\infty)}^2 \int_{\{|x| \geq R\}} \frac{|b_1(t)|}{1+|x|} + C \left\| \frac{b_2(t)}{1+|x|} \right\|_{L^\infty} \int_{\{|x| \geq R\}} p^2. \quad (2.27)$$

Note that as a function of  $t \in [0, T]$ , the right hand side of (2.27) is dominated by

$$C \left\| \frac{b_1(t)}{1+|x|} \right\|_{L^1} + C \|p\|_{L^\infty([0, T], L^2)}^2 \left\| \frac{b_2(t)}{1+|x|} \right\|_{L^\infty},$$

and the latter is an integrable function of  $t \in [0, T]$ . Furthermore, for a.e.  $t \in [0, T]$ , the right hand side of (2.27) tends to 0 as  $R \rightarrow \infty$ . Therefore by the dominated convergence theorem,

$$\lim_{R \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} p^2 \langle b, \nabla \phi_R \rangle = 0 \quad (2.28)$$

Next, in a similar way,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} p \langle \sigma^* \nabla p, \sigma^* \nabla \phi_R \rangle \right| &\leq \int_{\{R \leq |x| \leq 2R\}} |p| \cdot |\sigma^* \nabla p| \cdot \frac{1}{R} \|\nabla \phi\|_{L^\infty} |\sigma| \\ &\leq C \|\nabla \phi\|_{L^\infty} \int_{\{|x| \geq R\}} |p| \cdot |\sigma^* \nabla p| \frac{|\sigma|}{1+|x|}. \end{aligned}$$

By the assumptions on  $\sigma$ , we can split it into two matrix-valued functions  $\sigma_1$  and  $\sigma_2$ , such that

$$\frac{\sigma_1}{1+|x|} \in L^2([0, T], L^2(\mathbb{R}^n)) \quad \text{and} \quad \frac{\sigma_2}{1+|x|} \in L^2([0, T], L^\infty(\mathbb{R}^n)).$$

By Cauchy's inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} p \langle \sigma^* \nabla p, \sigma^* \nabla \phi_R \rangle \right| &\leq C \|p\|_{L^\infty} \int_{\{|x| \geq R\}} |\sigma^* \nabla p| \frac{|\sigma_1|}{1+|x|} + C \int_{\{|x| \geq R\}} |p| \cdot |\sigma^* \nabla p| \frac{|\sigma_2|}{1+|x|} \\ &\leq C \|\sigma^* \nabla p\|_{L^2} \left( \int_{\{|x| \geq R\}} \frac{|\sigma_1(t)|^2}{(1+|x|)^2} \right)^{1/2} \\ &\quad + C \left\| \frac{\sigma_2(t)}{1+|x|} \right\|_{L^\infty} \int_{\{|x| \geq R\}} |p| \cdot |\sigma^* \nabla p|. \end{aligned} \quad (2.29)$$

It is easy to see that the right hand side of (2.29) is dominated by

$$C\|\sigma^*\nabla p\|_{L^2}\left\|\frac{\sigma_1(t)}{1+|x|}\right\|_{L^2} + C\|p\|_{L^\infty([0,T],L^2)}\|\sigma^*\nabla p\|_{L^2}\left\|\frac{\sigma_2(t)}{1+|x|}\right\|_{L^\infty},$$

which, by Cauchy's inequality and the properties of  $\sigma_1, \sigma_2$  and  $p$ , is integrable with respect to  $t \in [0, T]$ . Moreover, for a.e.  $t \in [0, T]$ , the right hand side of (2.29) vanishes as  $R \rightarrow \infty$ . Consequently, the dominated convergence theorem leads to

$$\lim_{R \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} p \langle \sigma^* \nabla p, \sigma^* \nabla \phi_R \rangle = 0. \quad (2.30)$$

Finally it is clear that

$$-\int_{\mathbb{R}^n} p^2 \phi_R \operatorname{div}(b) \leq \int_{\mathbb{R}^n} p^2 \phi_R [\operatorname{div}(b)]^- \leq \|[\operatorname{div}(b)]^-\|_{L^\infty} \int_{\mathbb{R}^n} p^2. \quad (2.31)$$

Now we integrate the equation (2.26) in time from 0 to  $t$  (note that  $p(0) \equiv 0$ ) and obtain

$$\int_{\mathbb{R}^n} p^2 \phi_R \leq \int_0^T \int_{\mathbb{R}^n} p^2 \langle b, \nabla \phi_R \rangle - \int_0^T \int_{\mathbb{R}^n} p \langle \sigma^* \nabla \phi_R, \sigma^* \nabla p \rangle - \int_0^T \int_{\mathbb{R}^n} p^2 \phi_R \operatorname{div}(b).$$

By (2.28) and (2.30), for any  $\eta > 0$  there exists  $R_0 > 0$  such that for any  $R > R_0$ , we have

$$\left| \int_0^T \int_{\mathbb{R}^n} p^2 \langle b, \nabla \phi_R \rangle - \int_0^T \int_{\mathbb{R}^n} p \langle \sigma^* \nabla \phi_R, \sigma^* \nabla p \rangle \right| \leq \eta.$$

Taking into consideration the inequality (2.31), we get

$$\int_{\mathbb{R}^n} p^2 \phi_R \leq \eta + \int_0^T \|[\operatorname{div}(b)]^-\|_{L^\infty} \int_{\mathbb{R}^n} p^2.$$

First letting  $R \uparrow +\infty$  and then  $\eta \downarrow 0$ , we finally get

$$\int_{\mathbb{R}^n} p^2 \leq \int_0^T \|[\operatorname{div}(b)]^-\|_{L^\infty} \int_{\mathbb{R}^n} p^2.$$

Since  $[\operatorname{div}(b)]^- \in L^1([0, T], L^\infty(\mathbb{R}^n))$ , we have  $\int_{\mathbb{R}^n} p^2 = 0$ . This shows  $p = 0$  a.e. and the uniqueness is proved.  $\square$

**Remark 2.6.** Now we briefly discuss the reason why we are unable to deal with the diffusion coefficients  $\sigma$  of BV regularity. Recall the definition of  $S_\varepsilon$  in (2.5):

$$S_\varepsilon = [\rho_\varepsilon, \partial_i \sigma_{ik}](\sigma_{jk} \partial_j p) = \rho_\varepsilon * ((\sigma_{jk} \partial_j p) \partial_i \sigma_{ik}) - (\rho_\varepsilon * (\sigma_{jk} \partial_j p)) \partial_i \sigma_{ik}.$$

For simplicity, we assume that the functions are time-independent and denote by  $g = \sigma_{jk} \partial_j p \in L^2(\mathbb{R}^n)$ . Then

$$S_\varepsilon = \rho_\varepsilon * (g \partial_i \sigma_{ik}) - (\rho_\varepsilon * g) \partial_i \sigma_{ik}.$$

If  $\sigma_{ik}$  has only  $BV_{loc}$  regularity, then  $\mu := \partial_i \sigma_{ik}$  is a locally finite Radon measure on  $\mathbb{R}^n$ . We have the decomposition  $\mu = D^a \mu + D^s \mu$ , where  $D^a \mu \ll dx$  and  $D^s \mu \perp dx$ . In the case  $D^s \mu \neq 0$ , since  $g \in L^2(\mathbb{R}^n)$  is not continuous, the product  $g D^s \mu$  is sensitive to the modification of  $g$  in Lebesgue negligible sets. Therefore  $S_\varepsilon$  is not a well-defined object on  $\mathbb{R}^n$ .  $\square$

## References

- [1] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math. 158 (2004), 227–260.
- [2] L. Ambrosio, *Transport equation and Cauchy problem for non-Smooth Vector Fields*. Calculus of variations and nonlinear partial differential equations, 1–41, Lecture Notes in Math., 1927, Springer, Berlin, 2008.
- [3] L. Ambrosio and A. Figalli, *On flows associated to Sobolev vector fields in Wiener space: an approach à la DiPerna-Lions*. J. Funct. Anal. 256 (2009), no. 1, 179–214.
- [4] V.I. Bogachev, G. Da Prato and M. Röckner, *Fokker-Planck equations and maximal dissipativity for Kolmogorov operators with time dependent singular drifts in Hilbert spaces*. J. Funct. Anal. 256 (2009), no. 4, 1269–1298.
- [5] F. Cipriano and A.B. Cruzeiro, *Flows associated with irregular  $\mathbb{R}^d$ -vector fields*. J. Diff. Equations 210 (2005), 183–201.
- [6] G. Crippa and C. de Lellis, *Estimates and regularity results for the DiPerna-Lions flows*. J. Reine Angew. Math. 616 (2008), 15–46.
- [7] R.J. Di Perna and P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*. Invent. Math. 98 (1989), 511–547.
- [8] S. Fang, *ODE, SDE and PDE*. Adv. Math. (China) 38 (2009), 513–552.
- [9] S. Fang and D. Luo, *Transport equations and quasi-invariant flows on the Wiener space*. Bull. Sci. Math. 134 (2010), 295–328.
- [10] S. Fang, D. Luo and A. Thalmaier, *Stochastic differential equations with coefficients in Sobolev spaces*. J. Funct. Anal. 259 (2010), no. 5, 1129–1168.
- [11] A. Figalli, *Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients*. J. Funct. Anal. 254 (2008), 109–153.
- [12] C. Le Bris and P.L. Lions, *Renormalized solutions of some transport equations with partially  $W^{1,1}$  velocities and applications*. Ann. Mat. Pura Appl. 183 (2004), 97–130.
- [13] C. Le Bris and P.L. Lions, *Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients*. Comm. Partial Differential Equations 33 (2008), 1272–1317.
- [14] D. Luo, *Well-posedness of Fokker-Planck type equations on the Wiener space*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13 (2010), no. 2, 273–304.
- [15] M. Röckner and X. Zhang, *Weak uniqueness of Fokker-Planck equations with degenerate and bounded coefficients*. C. R. Math. Acad. Sci. Paris 348 (2010), no. 7–8, 435–438.
- [16] X. Zhang, *Stochastic flows of SDEs with irregular coefficients and stochastic transport equations*. Bull. Sci. Math. 134 (2010), 340–378.